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PHASE NOISE: THEORY - SPECIFICATION

by

NICHOLAS BRIENZA

Prepared for the

DEFENSE COMMUNICATIONS AGENCY

By the

COMPUTER SCIENCES CORPORATION

Under Contract No. DCA 100-70-C-0009

Task Order 0025, Part I

August 1970

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1.0 INTRODUCTION AND SUMMARY

The DSCS Modulation subsystem must interface with a channel composed of transmitting and receiving earth terminals, the uplink and downlink paths to and from the satellite, and the satellite, as in Figure 1.

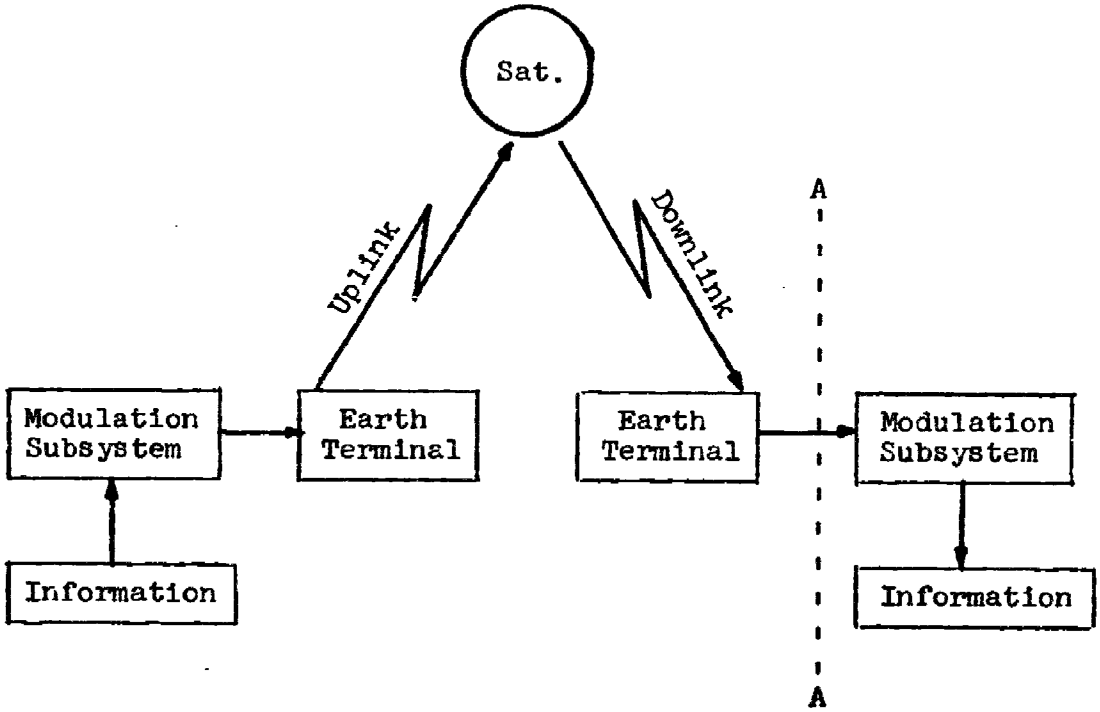


Figure 1. DSCS Modulation Subsystem Interface

The modulation subsystem is required to utilize phase shift key modulation to pass digital information over the satellite link and to recover this data at some specified error rate. In order to meet this requirement, the modulation subsystem employs coherent demodulation of the signal at interface A of Figure 1. However, coherent demodulation and thus the realizable error rate is dependent upon perfect carrier regeneration. Therefore, any errors in carrier tracking will subsequently degrade the achievable error rate. The major source of errors in carrier tracking results from the randomness of the carrier input to the modulation subsystem. Essentially, the system causes the carrier to become random and introduces additive thermal noise. In order to restrict induced corruption to the input signal, it is necessary to specify the carrier spectral purity.

This report discusses the subjects of spectral purity, imperfect carrier regeneration, and error rates. The various relationships required to identify the effects of the modulation subsystem input spectral purity on the realizable error rate are derived. Utilizing the relationships developed, a general specification of the spectral purity of the modulation subsystem input is defined.

Essentially, an approach to specifying system spectral purity requirements on the basis of a tolerable degradation to the probability of bit error has been developed.

2.0 SPECTRAL PURITY

The composite effects of many phenomena in the system cause the carrier to become a random process. Rather than isolate each of these phenomena, their composite effects may be viewed as if due to one source, "phase noise".

If a signal were to have all of its power concentrated at the carrier frequency, it would be spectrally pure. Phase noise causes the power in a signal to be spread out over frequencies other than that of the carrier. Thus, phase noise reduces the spectral purity of a signal.

There are various parameters that can be utilized as measures of spectral purity. Three of these parameters will be discussed in this section; Spectral Density, Incidental FM, and Frequency Stability.

2.1 SPECTRAL DENSITY

The power spectral density of a random process is a fundamental measure of spectral purity. It provides complete information about the frequency distribution of the average energy of a random process. Spectral density is defined[6] as in Equation (1).

$$S_n(\omega) = \int_{-\infty}^{\infty} R_n(\tau) e^{-j\omega\tau}\, d\tau \tag{1}$$

$$\omega = 2\pi f$$

where $R_n(\tau)$ = autocorrelation function of the r.p., n

Of major concern is the spectral purity of the carrier at the input to the modulation subsystem.

If this carrier is such that,

1. Its sidebands are orders of magnitude smaller than the carrier;
2. The AM power spectral density is negligible;
3. The mean square value of phase is much less than 1 rad^2,

its spectrum can be represented as [4]

$$S_{RF}(\omega) = P/2\,[S_\varphi(\omega + \omega_o) + S_\varphi(\omega - \omega_o)] \tag{2}$$

where P = carrier power

$S_{RF}(\omega)$ = two-sided power spectral density of carrier at input to modulation subsystem

$S_\varphi(\omega)$ = two-sided power spectral density of phase noise

ω_o = carrier frequency

It should be noted that Equation (2) holds for all frequencies except those within a small region of $\pm\omega_o$.

From Equation (2), it is obvious that only $S_\varphi(\omega)$, the phase noise power spectral density need be specified in order to specify the spectral purity of the modulation subsystem input carrier. Therefore, the subsequent discussions will center on $S_\varphi(\omega)$ rather than $S_{RF}(\omega)$.

2.2 INCIDENTAL FM

Incidental FM is a common term, often used to describe the effect of phase noise. Typically, the parameters Δf, the rms frequency deviation measured in a band of "B" Hz, and f_m, the modulating frequency associated with Δf, are employed in defining incidental FM. It can be shown (see Appendix A) that Δf, f_m and $S_\varphi(\omega)$ are related as in Equation (3).

$$\Delta f^2 = 4B\ f_m^{\ 2} S_\varphi(2\pi f_m) \qquad (3)$$

Since only positive values of Δf are meaningful, Equation (3) indicates that Δf is a unique function of $S_\varphi(\omega)$. Therefore, a specification of $S_\varphi(\omega)$ adequately suffices as a specification of Δf and f_m.

2.3 FREQUENCY STABILITY

Frequency stability represents a third approach to describing the effects of phase noise. The parameter most consistently utilized to define frequency stability is the average fractional frequency departure of Equation (4).[4]

$$\frac{\sigma^2\left[\langle\dot{\varphi}\rangle_{t,\tau}\right]}{\omega_o^{\ 2}} = \frac{2}{\pi(\omega_o\tau)^2} \int_{-\infty}^{\infty} S_\varphi(\omega)\ \sin^2 \omega\tau/2\ d\omega \qquad (4)$$

ω_o = reference frequency

τ = averaging time

$\langle\dot{\varphi}\rangle_{t,\tau}$ = time average, over a period τ centered at t, τ of the frequency departure, $\dot{\varphi}$

It should be noted that Equation (4) is based on the assumption that $\dot{\varphi}$ exists for all time. In actual practice however, the value of $\sigma^2[\langle\dot{\varphi}\rangle_{t,\tau}]$ must be determined on the basis of a finite number of samples of $\langle\dot{\varphi}\rangle_{t,\tau}$. Allan[9] has shown that under the conditions of a finite number of samples the value of $\sigma^2[\langle\dot{\varphi}\rangle_{t,\tau}]$ can be dependent on the following parameters:

N = number of samples of $\langle\dot{\varphi}\rangle_{t,\tau}$

T = period of sampling

τ = sample time

ω_B = measurement system bandwidth

These parameters are directly related to the measurement technique utilized to evaluate the average fractional frequency departure. Nevertheless, the fundamental parameter in determining $\sigma^2[\langle\dot{\varphi}\rangle_{t,\tau}]$ is still the power spectral density, $S_\varphi(\omega)$ which is independent of the measurement technique employed.

Since both incidental FM and frequency stability are unique functions of the phase noise power spectral density, it is only reasonable that $S_\varphi(\omega)$ should be utilized as the fundamental parameter for specifying spectral purity. It will also be shown in subsequent sections that $S_\varphi(\omega)$ is a useful parameter in the analysis of carrier tracking errors. Another important feature of $S_\varphi(\omega)$ is that it is commonly used in industry as a measure of spectral purity.

2.4 PHASE NOISE SPECTRAL CHARACTERIZATION

Phase noise may be characterized as a phenomena which essentially spreads out the power in a carrier. That is, the carrier no longer has a discrete line power spectrum but a continuous power spectral density. There are several types of phase noise each easily characterized by its power spectral density as in Table I.

TABLE I. PHASE NOISE CHARACTERIZATION

Type of Phase Noise	Power Spectral Density
1) Random walk of frequency	$S_\varphi(\omega) = h_4/\omega^4$
2) Flicker of frequency	$S_\varphi(\omega) = h_3/\lvert\omega^3\rvert$
3) White frequency	$S_\varphi(\omega) = h_2/\omega^2$
4) Flicker of phase	$S_\varphi(\omega) = h_1/\lvert\omega\rvert$
5) White phase	$S_\varphi(\omega) = h_0$

It should be noted that the h_i of Table I are constants.

Typically the carrier at the modulation subsystem input will have a spectrum made up of one or more of the five types of phase noise in Table I. For instance, flicker of frequency might dominate in a region close to the carrier and white phase might dominate further away from the carrier. In general, a model of the carrier spectrum can be constructed from the typesof phase noise listed in Table I.

2.5 ADDITIVE WHITE GAUSSIAN NOISE

The previous discussions in this section have centered on the spectral density of the carrier. However, the actual signal with which the modulation subsystem must deal consists of a carrier component and a noise component. The noise component is typically additive white Gaussian noise. Since the carrier and the additive noise are statistically independent, their composite power spectral density is just the sum of the power spectral density of each. Therefore, using Equation (2) a composite spectral density can be defined as in Equation (5).

$$S_c(\omega) = S_{RF}(\omega) + N_o/2 \tag{5}$$

where $S_c(\omega)$ = two-sided spectral density of carrier plus white noise

N_o = one-sided spectral density of white noise

$S_{RF}(\omega)$ = is as defined in Equation (2).

2.6 MEASUREMENT OF PHASE NOISE SPECTRAL DENSITY

There are two commonly employed approaches to the measurement of the spectral density of phase noise.

The first approach involves passing the corrupted carrier plus noise through an ideal phase detector. The output of the phase detector is then spectrally analyzed. This technique is common in the oscillator and frequency synthesizer industry.

A second approach involves direct spectral analysis (B Hz band) of the corrupted carrier plus noise. It should be noted that this technique is meaningful only under the conditions for which Equation (2) holds (see Section 2.1). The conditions for Equation (2) essentially insure that the carrier spectrum sidebands are of the same shape as the phase noise spectrum.

$$S_{RF}(\omega) = P/2\ [S_\varphi(\omega + \omega_o) + S\ (\omega - \omega_o)] \tag{2}$$

Both approaches provide measures of the phase noise power spectral density $S_\varphi(\omega)$. This information is typically displayed on a phase noise power to carrier power ratio vs offset frequency plot as in Figure 2. The phase noise power to carrier power ratio for approach 2 is related to the phase noise power spectral density as in Equation (5a) (see Appendix B).

$$\frac{N_p}{P} = S_\varphi(2\pi f_m)B + \frac{N_o}{P} B \qquad (5a)$$

where B = band in which power is measured in Hz.

It should be noted that Equation (5a) holds only for small values of B.

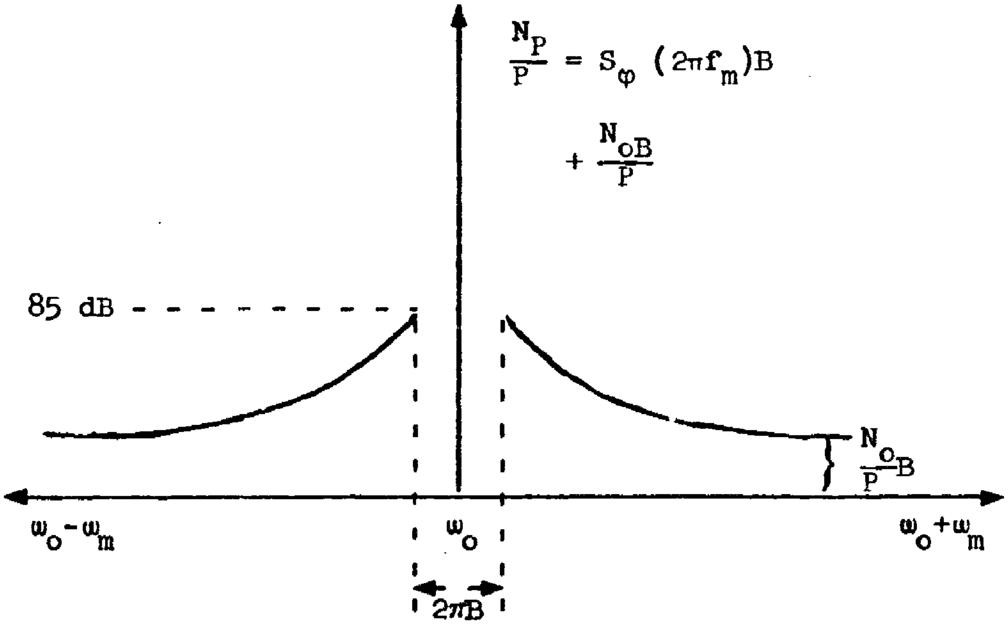


Figure 2. Phase Noise Power to Carrier Power Ratio vs Offset Frequency ω_m as Measured in a B Hz Band

The major disadvantage inherent in both measurement approaches is that the spectrum cannot be identified in a region B Hz wide around the carrier, However, as long as

$B \ll f_n$

f_n = natural frequency of the carrier tracking loop

this region of the spectrum contributes negligibly to the carrier tracking error. Therefore, it does not constitute a major obstacle. Nevertheless, the region of the spectrum close to the carrier does determine the long term action of the carrier. Therefore, long term frequency stability for pertinent averaging times should be specified. The subject of long term frequency stability will be dealt with in a subsequent report. A more detailed discussion of exact measurements techniques will also appear in a subsequent report.

3.0 DEGRADATION IN ERROR RATE

Coherent demodulation is employed in the modulation subsystem to insure that digital information is recovered at the lowest possible error rate. However, coherent demodulation and thus the realizable error rate are dependent on perfect carrier tracking. Therefore, any errors in the carrier tracking loops will tend to increase the probability of error of the recovered data bits. In order to determine the degradation due to carrier tracking errors, a general relationship between the probability of error and the carrier tracking error must be derived.

3.1 RELATIONSHIP BETWEEN PHASE ERROR AND PROBABILITY OF ERROR

The exact relationship between the conditional probability of error and the carrier tracking error due to a Costas loop appears in Equation (6)[1]

$$P_e(\varphi_e) = \mathrm{Erfc}[\sqrt{s}\cos\varphi_e] \tag{6}$$

where $P_e(\varphi_e)$ = probability of error given the phase error φ_e

$s = \frac{ST_s}{N_o}(1-\lambda)$

λ = signal cross correlation coefficient

S = signal power

T_s = duration of symbol bit

N_o = one-sided spectral density of the input white Gaussian noise

Equation (6) can be rewritten for the case of antipodal PSK modulation to give,

$$P_e(\varphi_e) = \mathrm{Erfc}\left[\sqrt{\frac{2E_s}{N_o}\cos^2\varphi_e}\right] \tag{7}$$

where $E_s = ST_s$ = energy per symbol.

It should be noted that φ_e in Equation (7) is a random variable. Therefore, the conditional probability of error is also random. It can be shown that if the variance of φ_e is relatively small, the average value of $P_e(\varphi_e)$ is defined by Equation (8).[7]

$$P_e = E\{P_e(\varphi_e)\} = \mathrm{Erfc}\left[\sqrt{\frac{2E_s}{N_o}\left(1 - \sigma_e^2\right)}\right] \tag{8}$$

where P_e = average probability of data bit error

σ_e^2 = variance of the carrier tracking errors

The factor $(1 - \sigma_e^2)$ in Equation (8) reduces the effective value of E_s/N_o and thus increases the probability of error. Therefore, a degradation factor due to imperfect carrier regeneration can be defined as follows,

$$L = -10 \log_{10} (1 - \sigma_e^2) \text{ dB} \tag{9}$$

L as defined in Equation (9) indicates in dB the reduction in E_s/N_o due to imperfect carrier regeneration.

4.0 EFFECTS OF SPECTRAL PURITY ON ERROR RATE

In Section 2, the power spectral density was identified as a fundamental measure of carrier plus noise. In Section 3, the variance of the carrier tracking error was indicated as a source of degradation in the probability of error. In this section a relationship between the composite carrier plus noise power spectral density and the mean square carrier tracking error will be derived. From this relationship the effects of spectral purity on error rate can be determined. Throughout the discussion, the tracking loop error will be required to be small and of zero mean. Thus, a linear analysis of the Costas Loop may be employed.

4.1 MEAN SQUARE CARRIER TRACKING ERROR

Utilizing linear system theory for random processes the mean square carrier tracking error due to phase noise can be shown to be

$$\sigma_e^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\varphi(\omega) \left|1 - H(j\omega)\right|^2 d\omega \text{ rad}^2 \tag{10}$$

where σ_e^2 = mean square carrier tracking error due to phase noise

$S_\varphi(\omega)$ = power spectral density of phase noise

$H(\omega)$ = closed loop transfer function of linear phase lock loop theory

Another source of tracking error that is of significance in the DSCS is doppler rate. Utilizing linear system theory it can be shown that the mean square carrier trading error due to doppler rate $\dot{f}_d$ is[3]

$$\sigma_{De}^{\ 2} = \left[\frac{2\pi\dot{f}_d}{\omega_n^{\ 2}}\right]^2 \quad \text{rad}^2 \tag{11}$$

where ω_n = natural frequency of the 2^{nd} order loop of linear phase lock loop theory

$\dot{f}_d$ = doppler rate in Hz/sec^2

The third source of carrier tracking error is additive white Gaussian noise. It can be shown[3] that the mean square error due to additive white noise is

$$\sigma_n^{\ 2} = \frac{N_o}{2P} \cdot \frac{1}{2\pi} \int_{-2\pi R_s}^{2\pi R_s} \left|H(j\omega)\right|^2 d\omega$$

or

$$\sigma_n^{\ 2} \approx \frac{N_o}{2P} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|H(j\omega)\right|^2 d\omega \quad \text{rad}^2 \tag{12}$$

N_o = one-sided noise density

P = carrier power

$H(j\omega)$ = closed loop transfer function of linear phase lock loop theory

Equation (12) may be rewritten as

$$\sigma_n^{\ 2} = \frac{N_o}{2E_s} \cdot \frac{1}{R_s} \int_{-\infty}^{\infty} \left|H(j\omega)\right|^2 d\omega \quad \text{rad}^2 \tag{12a}$$

The noise density of Equation (12a) should be modified to reflect the Costas Loop implementation loss. Essentially the removal of the data by the Costas Loop degrades carrier tracking by raising the thermal noise density.[8] The degradation factor $[1 + N_o/E_s]$ is included in Equation (12b).

$$\sigma_n^2 = \frac{N_o\left[1 + \frac{N_o}{E_s}\right]}{2E_s R_s} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|H(j\omega)\right|^2 d\omega \quad \text{rad}^2 \qquad (12b)$$

where the data rate filters in the Costas Loop have low pass bandwidths of R_s Hz.

4.1.1 Total Mean Square Carrier Tracking Error

Three sources of carrier tracking error have been identified. The manner in which these sources contribute to the total mean square error should be considered. Recall, it was required that the phase error in the Costas Loop be small. Therefore, the loop is linear. Since the carrier and the additive noise are uncorrelated, the total mean square tracking error is

$$\sigma_{Te}^2 = \sigma_n^2 + \sigma_{De}^2 + \sigma_e^2 \quad \text{rad}^2 \qquad (13)$$

where σ_{De}^2 = mean square error due to doppler rate
σ_e^2 = mean square error due to phase noise
σ_n^2 = mean square error due to additive white noise

4.2 RELATIONSHIP BETWEEN SPECTRAL DENSITY AND PROBABILITY OF ERROR

In order to relate composite spectral density and the probability of error, it is necessary to decide on a form for the phase noise spectral density. If crystals are employed in the system, the dominant type of phase noise close to the carrier is flicker of frequency[4,9]. However, if frequency synthesizers are employed, the dominant type of phase noise close to the carrier is flicker of phase[10]. In either case, a model of the phase noise power spectral density can be constructed as in Figure 3.

There are various points on Figure 3 that require further explanation. Note that the spectrum is terminated at $2\pi R_s$. The carrier tracking Costas Loop has two data rate filters that preclude the spectrum beyond $2\pi R_s$ from having any significant effect on carrier tracking. The spectrum is not defined in a region $2\pi B$ radians/sec around the carrier. As explained in Section 2.6, this

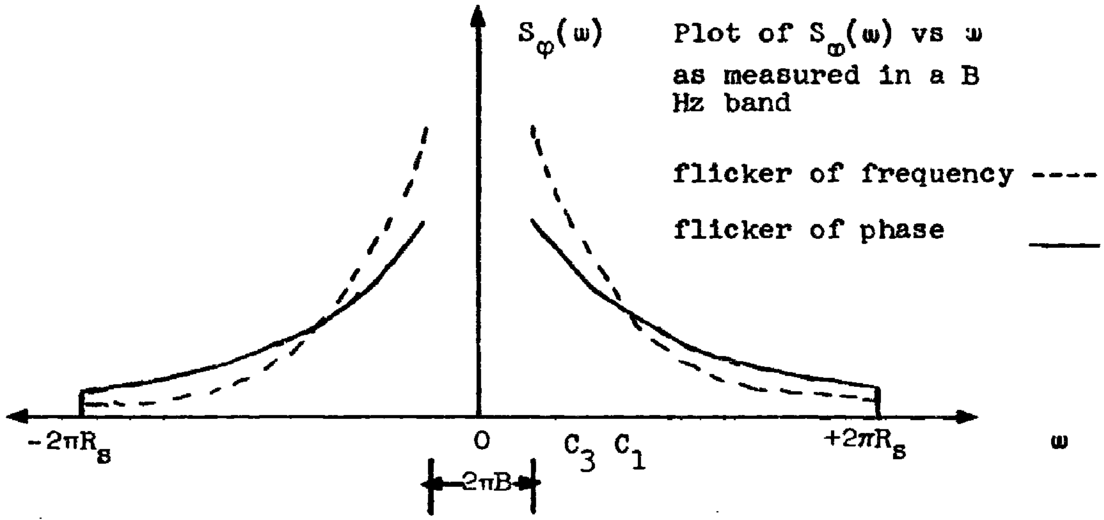


Figure 3. Model of Phase Noise Power Spectral Density

is due to the resolution capability of measurement techniques. However, if $B \ll f_n$ (the loop natural frequency), this region contributes negligibly to carrier tracking error.

It should be noted that if relatively clean crystals and frequency synthesizers are employed for frequency translations in a system, the two spectra of Figure 3 would be added to produce a composite phase noise spectrum. If the crystals and synthesizers were quite noisy, the composite spectrum would be the convolution of the two spectra.

Utilizing the spectral model of Figure 3 and Equation (10), a composite mean square tracking error due to phase noise can be calculated.

$$\sigma_{ec}^2 = \frac{1}{2\pi}\int_{-2\pi R_s}^{2\pi R_s} \frac{h_1}{|\omega|}\left|1 - H(j\omega)\right|^2 d\omega \;\delta(1)$$

$$+ \frac{1}{2\pi}\int_{-2\pi R_s}^{2\pi R_s} \frac{h_3}{|\omega^3|}\left|1 - H(j\omega)\right|^2 d\omega\; \delta(k)\,\mathrm{rad}^2 \qquad (14)$$

where σ_{ec}^2 = composite mean square error due to phase noise

$\delta(\cdot)$ = Kronecker delta

In Equation (14), if k = 0 and i = 0, both flicker of frequency and phase are present.

By substituting Equations (11), (12) and (14) into Equation (13) the composite mean square tracking error can be determined.

$$\sigma_{Te}^{2} = \frac{1}{2\pi}\int_{-2\pi R_s}^{2\pi R_s}\left[\frac{h_1}{|\omega|}\,\delta(i) + \frac{h_3}{|\omega^3|}\,\delta(k)\right]\left|1-H(j\omega)\right|^2 d\omega$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N_o\left[1+\frac{N_o}{E_s}\right]}{2E_sR_s}\left|H(j\omega)\right|^2 d\omega + \left[\frac{2\pi\dot{f}_d}{\omega_n^{\,2}}\right]^2 \text{ rad}^2 \qquad (15)$$

Note that Equation (15) is a function of h_1, h_3, E_s/N_o, R_s, $\dot{f}_d$, ω_n, and H(s). System requirements dictate the values of $\dot{f}_d$ and E_s/N_o. H(s) is required to be a second order loop transfer function, thus, it is defined if ζ, the damping factor, and ω_n, the natural frequency, are indicated. In order to simplify the analysis, a ζ of 0.707 will be employed.

The contribution to the total mean square error due to phase noise is evaluated in Appendix C. Equation (15) may be rewritten as

$$\sigma_{Te}^{2} = \frac{N_o\left[1+\frac{N_o}{E_s}\right]}{2E_sR_s}\cdot(1.06\,\omega_n) + \left[\frac{2\pi\dot{f}_d}{\omega_n^{\,2}}\right]^2$$

$$+ \frac{h_1\delta(i)}{4\pi}\ln\left[\frac{\omega_n^{\,4}+(2\pi R_s)^4}{\omega_n^{\,4}+(\pi B)^4}\right] + \frac{h_3\delta(k)}{2\pi\omega_n^{\,2}}\left[\tan^{-1}\frac{(2\pi R_s)^2}{\omega_n^{\,2}}\right. \qquad (16)$$

$$\left. - \tan^{-1}\frac{(\pi B)^2}{\omega_n^{\,2}}\right] \text{ rad}^2$$

Recall in Section 2.6, it was required that B << f_n, therefore Equation (16) becomes,

$$\sigma_{Te}^{2} = \frac{N_o\left[1 + \frac{N_o}{E_s}\right]}{2E_sR_s}(1.06\ \omega_n) + \left[\frac{2\pi\dot{f}_d}{\omega_n^{2}}\right]^2$$

$$+ \frac{h_1\delta(i)}{4\pi}\ \ln\left[\frac{\omega_n^{4} + (2\pi R_s)^4}{\omega_n^{4}}\right] \tag{17}$$

$$+ \frac{h_3\delta(k)}{2\pi\omega_n^{2}}\ \tan^{-1}\left(\frac{2\pi R_s}{\omega_n}\right)^2$$

Let's define a parameter γ, the ratio of the data rate to the loop natural frequency.

$$\gamma = R_s/f_n \tag{18}$$

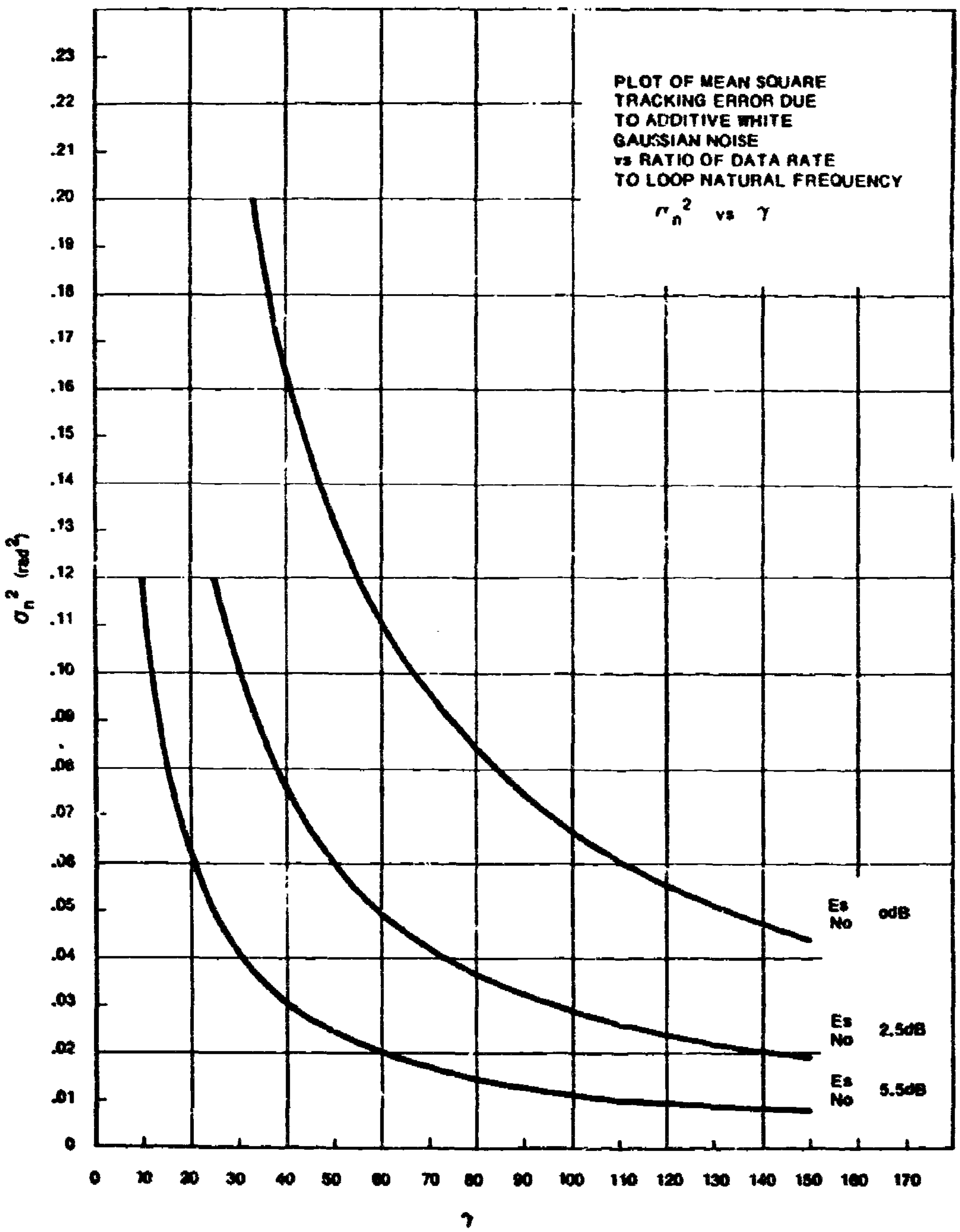
Using Equations (18) and (17), the total mean square error can be written as

$$\sigma_{Te}^{2} = \left(\frac{N_o\left[1 + \frac{N_o}{E_s}\right]}{2E_s}\right)\cdot\frac{6.66}{\gamma} + \left[\frac{\dot{f}_d\gamma^{2}}{2\pi R_s^{2}}\right]^2$$

$$+ \frac{h_1\delta(i)}{4\pi}\ \ln\left[1 + \gamma^4\right] + \frac{h_3\gamma^{2}\delta(k)}{(2\pi)^3R_s^{2}}\ \tan^{-1}(\gamma^2) \tag{19}$$

Equation (19) can be broken into its four parts and the values of each part can be evaluated separately. Plots of the various mean square errors as functions of γ appear in Graphs 1-4.

Graphs 1, 2, 3, and 4 can be utilized to derive a composite spectrum specification for the modulation system input. Appendix D provides an example of such a spectrum derivation. Appendix E derives the degradation due to phase noise of a common oscillator. As a reference, a short term stability measurement for the same oscillator is also given.


PLOT OF MEAN SQUARE
TRACKING ERROR DUE
TO ADDITIVE WHITE
GAUSSIAN NOISE
vs RATIO OF DATA RATE
TO LOOP NATURAL FREQUENCY
σ_n^2 vs γ
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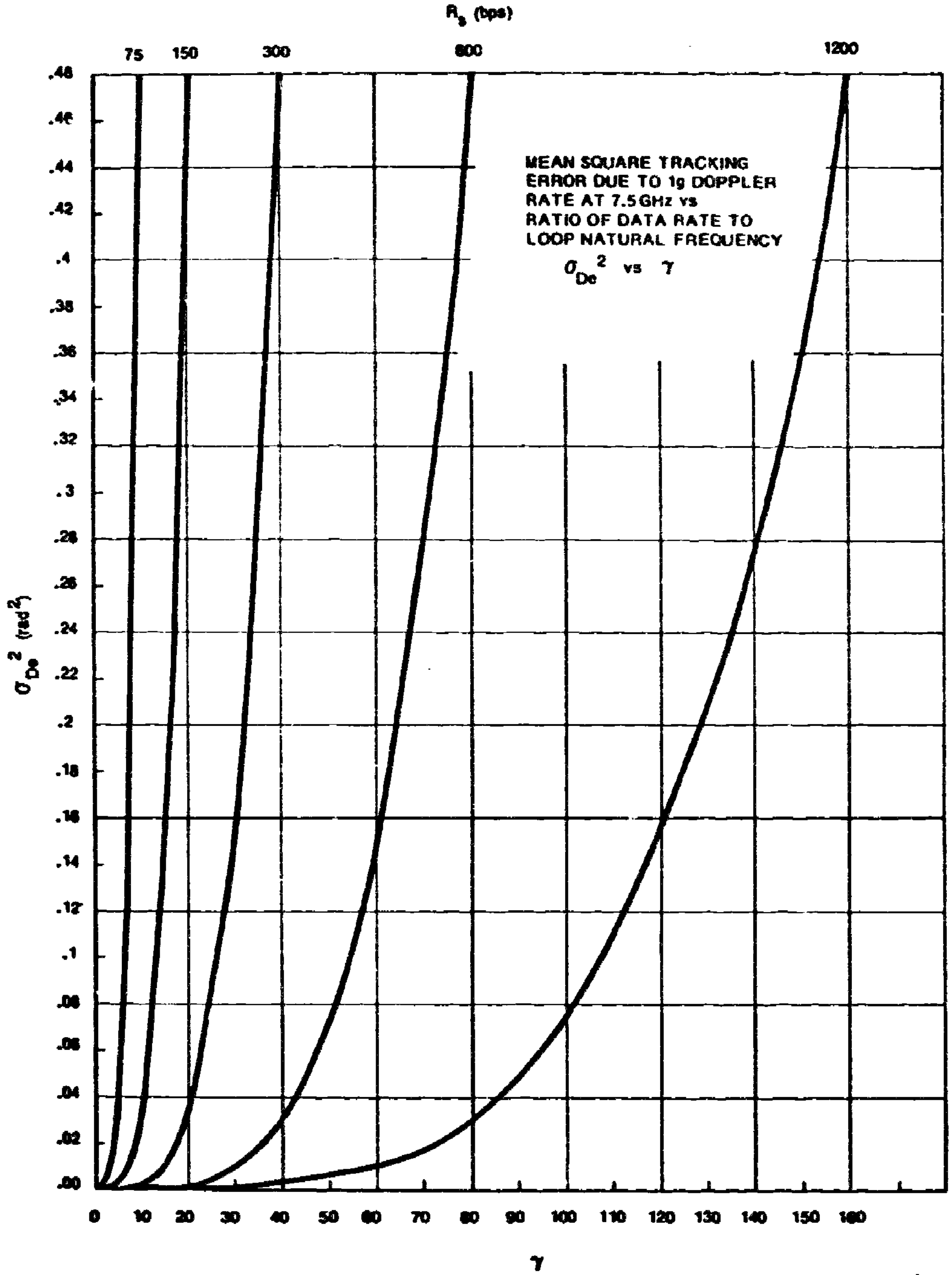
Graph 1



R_s (bps)
75
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300
600
1200
MEAN SQUARE TRACKING
ERROR DUE TO 1g DOPPLER
RATE AT 7.5 GHz vs
RATIO OF DATA RATE TO
LOOP NATURAL FREQUENCY
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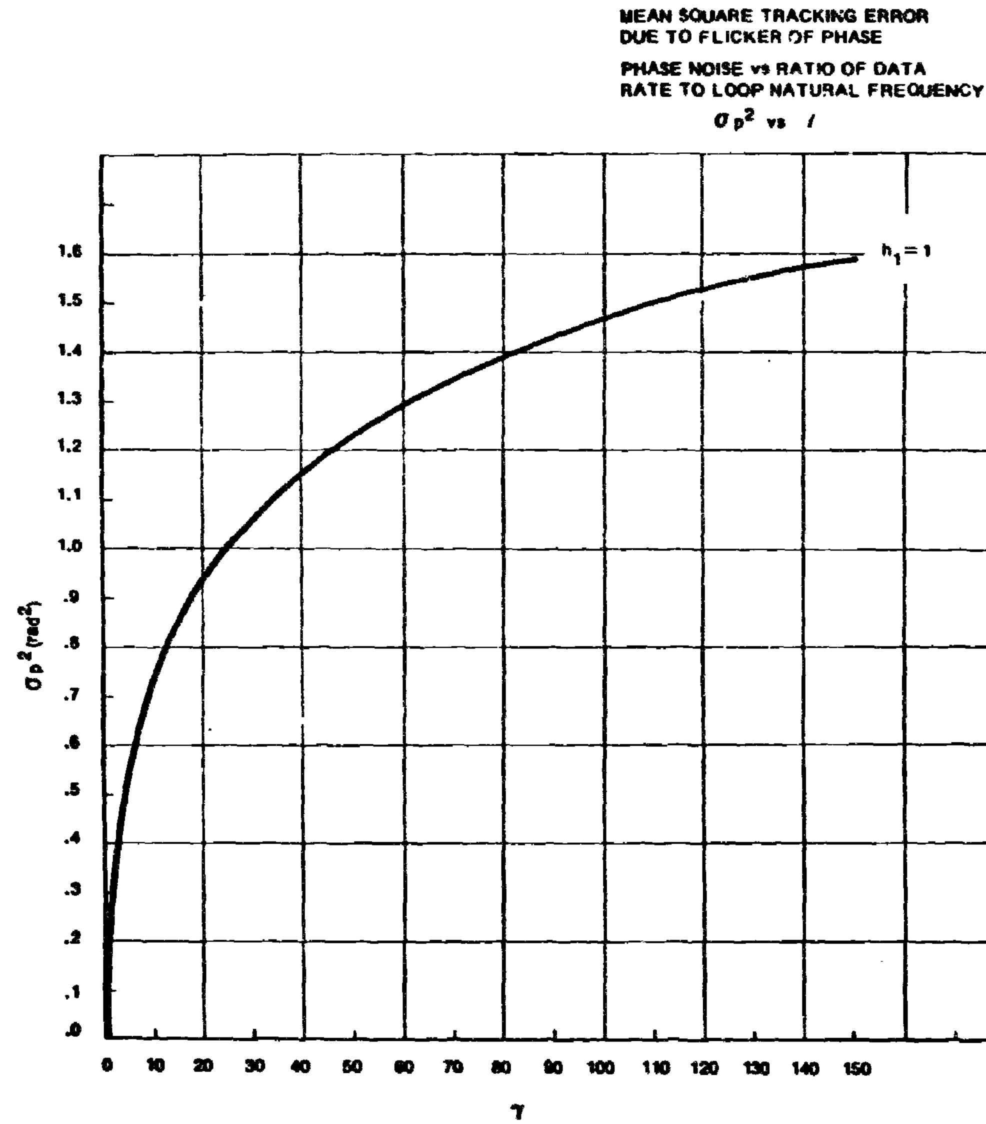
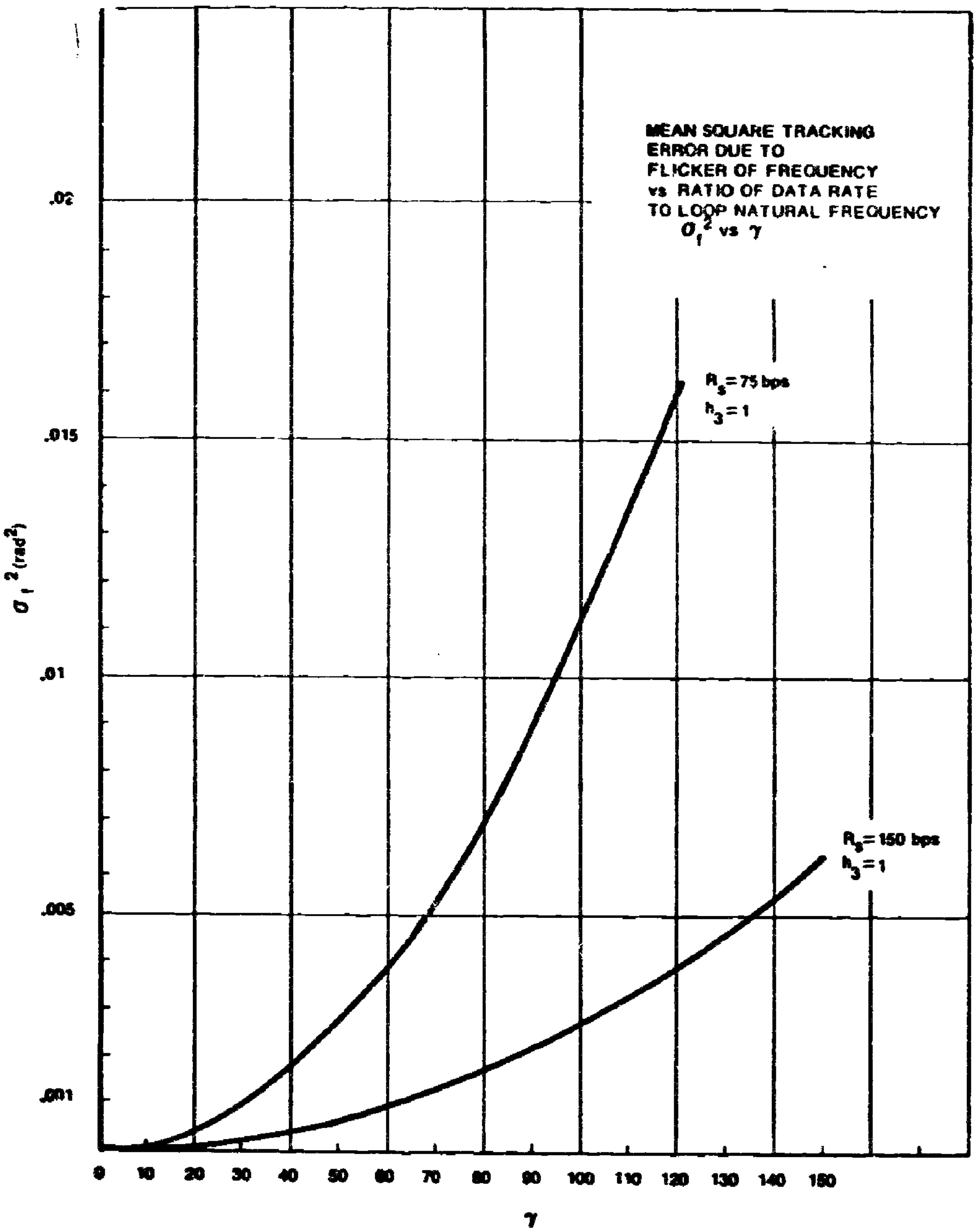
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
γ

Graph 2


MEAN SQUARE TRACKING ERROR
DUE TO FLICKER OF PHASE
PHASE NOISE vs RATIO OF DATA
RATE TO LOOP NATURAL FREQUENCY
σ_p^2 vs γ
$h_1 = 1$
σ_p^2 (rad^2)
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
.0
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150
γ

Graph 3



MEAN SQUARE TRACKING ERROR DUE TO FLICKER OF FREQUENCY vs RATIO OF DATA RATE TO LOOP NATURAL FREQUENCY
σ_f^2 vs γ
R_s = 75 bps
h_3 = 1
R_s = 150 bps
h_3 = 1
σ_f^2 (rad^2)
.02
.015
.01
.005
.001
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
γ

Graph 4

ACKNOWLEDGEMENT

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APPENDIX A

DERIVATION OF RELATIONSHIP BETWEEN PHASE NOISE POWER SPECTRAL DENSITY $S_{\varphi}(\omega)$ AND THE FM MODULATION INDEX β

If a signal is such that its sidebands are orders of magnitude smaller than the carrier, the AM power spectral density is negligible, and the mean square value of phase is much less than 1 rad^2, the spectrum of the composite signal can be represented as [4]

$$S_{RF}(\omega) = P/2\,[S_{\varphi}(\omega + \omega_o) + S_{\varphi}(\omega - \omega_o)] \qquad \text{(A-1)}$$

where P = carrier power

$S_{RF}(\omega)$ = composite signal spectrum

$S_{\varphi}(\omega)$ = spectrum of phase noise

ω_o = carrier frequency

Equation (A-1) holds for all frequencies except those within a small region around $\pm\omega_o$.

If the power in a B Hz band, f_m Hz from the carrier frequency ω_o were measured, its value would be

$$P_{fm} = \frac{1}{2\pi}\int_{\omega_o+2\pi(f_m-B/2)}^{\omega_o+2\pi(f_m+B/2)} 2S_{RF}\,(\omega)\;d\omega \qquad \text{(A-2)}$$

If B is very small, $S_{RF}(\omega)$ is essentially constant over the interval. Therefore Equation (A-2) becomes

$$P_{fm} = 2B\; S_{RF}\,(\omega_o + 2\pi f_m) \qquad \text{(A-3)}$$

Substituting Equation (A-1) into (A-3) and recognizing that only $S_{\varphi}(\omega-\omega_o)$ contributes to the integral of (A-2), Equation (A-4) results

$$P_{fm} = P\cdot B\cdot S_{\varphi}(2\pi f_m) \qquad \text{(A-4)}$$

Phase noise may also be viewed as a phenomena that essentially narrowband frequency modulates the carrier. Therefore, the power in a BHz band, f_m Hz from the carrier can be approximated as[5]

$$P_{fm} = (\beta^2/4)\, P \tag{A-5}$$

Combining Equations (A-4) and (A-5), $S_\varphi(\omega)$ can be defined as a function of β.

$$\frac{\beta^2}{4B} = S_\varphi(2\pi f_m) \tag{A-6}$$

If $S_\varphi(\omega)$ were a one-sided spectral density $S'_\varphi(\omega)$ Equation (A-6) would become

$$\frac{\beta^2}{2B} = S'_\varphi\,(2\pi f_m) \tag{A-7}$$

The modulation index β is defined as

$$\beta = \frac{\Delta f}{f_m} \tag{A-8}$$

where Δf = peak frequency deviation

f_m = frequency of modulation

Substituting (A-8) into (A-6), we derive

$$\Delta f^2 = 4Bf_m{}^2\, S_\varphi(2\pi f_m) \tag{A-9}$$

Equation (A-9) demonstrates that Δf is a unique function of $S_\varphi(\omega)$ (only positive values of Δf have significance).

APPENDIX B

RELATIONSHIP BETWEEN PHASE NOISE POWER-TO-CARRIER POWER RATION AND PHASE NOISE SPECTRAL DENSITY

The following analysis is valid when the carrier spectrum sidebands are of the same shape as the phase noise spectrum (see Equation (2)).

Phase noise power at f_m Hz from the carrier is determined by measuring the power in a BHz band f_m Hz from the carrier.

Using Equation (5) this same power can be calculated as in Equation (B-1).

$$N_p\Big|_{BHz} = \frac{2}{2\pi} \int_{2\pi(f_m-B/2)}^{2\pi(f_m+B/2)} S_{RF}(\omega)\, d\omega + \frac{2}{2\pi} \int_{2\pi(f_m-B/2)}^{2\pi(f_m+B/2)} N_o/2\, d\omega \qquad \text{(B-1)}$$

Substituting Equation (2) into (B-1) we get

$$N_p\Big|_{BHz} = \frac{2}{2\pi} \int_{\omega_o+2\pi(f_m-B/2)}^{\omega_o+2\pi(f_m+B/2)} P/2[S_\varphi(\omega + \omega_o) + S_\varphi(\omega - \omega_o)]\, d\omega \qquad \text{(B-2)}$$

$$+ BN_o$$

If B is relatively small, $S_\varphi(\omega)$ is essentially constant over the region of integration. Therefore, Equation (B-2) becomes

$$N_p\Big|_{BHz} = PS_\varphi(2\pi f_m)B + BN_o \qquad \text{(B-3)}$$

If B is 1 Hz, Equation (B-3) becomes

$$N_p\Big|_{BHz} = PS_\varphi(2\pi f_m) + N_o \qquad \text{(B-4)}$$

Recognizing that the carrier power is P, the ratio of phase noise power to carrier power becomes

$$\frac{N_p}{P} = S_{\varphi}(2\pi f_m)B + \frac{N_o}{P} B \tag{B-5}$$

N_o = One sided noise density

P = Carrier power

B = Bandwidth of measurement in Hz

APPENDIX C

EVALUATION OF MEAN SQUARE TRACKING ERROR DUE TO PHASE NOISE

From Equation (14) of the text, the composite mean square error due to phase noise is

$$\sigma_{ec}^{2} = \frac{1}{2\pi} \int_{-2\pi R_s}^{2\pi R_s} \left[\frac{h_1}{|\omega|} \delta(i) + \frac{h_3}{|\omega^3|} \delta(k) \right] \left| 1-H(j\omega) \right|^2 d\omega \qquad \text{(C-1)}$$

where $\delta(\cdot)$ is the Kronecker delta

Evaluate each integral separately. The error due to flicker of phase for a loop with $\zeta = .707$ is

$$\sigma_{ep}^{2} = \frac{1}{\pi} \int_{\pi B}^{2\pi R_s} \frac{h_1}{\omega} \frac{\omega^4}{\omega_n^{4} + \omega^4} d\omega$$

$$= \frac{h_1}{\pi} \left[\tfrac{1}{4} \ln \left[\omega_n^{4} + \omega^4 \right] \right] \Bigg|_{-\pi B}^{2\pi R_s}$$

$$\sigma_{ep}^{2} = \frac{h_1}{4\pi} \left[\ln \frac{\omega_n^{4} + (2\pi R_s)^4}{\omega_n^{4} + (\pi B)^4} \right] \qquad \text{(C-2)}$$

The error due to flicker of frequency for a loop with $\zeta = .707$ is

$$\sigma_{ep}^{2} = \frac{1}{\pi} \int_{-\pi B}^{2\pi R_s} \frac{h_3}{\omega^3} \frac{\omega^4}{\omega_n^{4} + \omega^4} d\omega$$

$$= \frac{h_3}{2\pi \omega_n^{2}} \tan^{-1} \frac{\omega^2}{\omega_n^{2}} \Bigg|_{\pi B}^{2\pi R_s}$$

$$\sigma_{ep}^{2} = \frac{h_3}{2\pi\omega_n^{2}} \left[\tan^{-1} \frac{(2\pi R_s)^2}{\omega_n^{2}} - \tan^{-1} \frac{(\pi B)^2}{\omega_n^{2}} \right] \qquad \text{(C-3)}$$

APPENDIX D

DERIVATION OF PHASE NOISE SPECIFICATION FOR PHASE II

CALCULATION OF OPTIMUM LOOP BANDWIDTH

For a loop that must operate in the presence of thermal noise and acceleration, the total mean square tracking error is

$$\sigma_{T_e}{}^2 = \sigma_{D_e}{}^2 + \sigma_n{}^2 \qquad \text{(D-1)}$$

$$= B/\omega_n{}^4 + A\omega_n$$

where

$B = [2\pi\dot{f}d]^2$; $\sigma_{D_e}{}^2$ = mean square tracking error due to doppler rate (acceleration)

$A = \dfrac{1.06\ N_o}{2E_sR_s}\ [1 + N_o/E_s]$; $\sigma_n{}^2$ = mean square tracking error due to thermal noise.

Minimize $\sigma_{T_e}{}^2$ with respect to ω_n

$$\frac{d\sigma_{T_e}{}^2}{d\omega_n} = -4B\omega_n{}^{-5} + A = 0$$

$$4B\omega_n{}^{-5} = A$$

$$\omega^5{}_n = \frac{4B}{A}$$

$$\omega_n = \left(\frac{4B}{A}\right)^{1/5} \qquad \text{(D-2)}$$

Equation (D-2) gives the optimum value for the loop natural frequency. Substituting Equation (D-2) into Equation (D-1), the minimum value of the total mean square tracking error can be calculated.

$$\sigma_{T_e}^{2}\,min = \frac{B}{\left(\frac{4B}{A}\right)^{4/5}} + A\left(\frac{4B}{A}\right)^{1/5}$$

$$= \frac{B + A\left(\frac{4B}{A}\right)}{\left(\frac{4B}{A}\right)^{4/5}}$$

$$\sigma_{T_e}^{2}\,min = \frac{5B}{\left(\frac{4B}{A}\right)^{4/5}} = 5\,\sigma_{D_e}^{2}\,min \qquad \text{(D-3)}$$

From Equation (D-3) the value of the minimum mean square tracking error due to doppler is

$$\sigma_{D_e}^{2}\,min = \frac{1}{5}\,\sigma_{T_e}^{2}\,min \qquad \text{(D-4)}$$

Equation (D-4) can be rewritten using Equation (D-1)

$$\frac{B}{\omega_{n\,min}^{4}} = \frac{1}{5}\,\sigma_{T_e}^{2}\,min \qquad \text{(D-5)}$$

Solving Equation (D-5) for $\omega_{n\,min}$,

$$\omega_{n\,min} = \left(\frac{5B}{\sigma_T^{2}}\right)^{1/4} \qquad \text{(D-6)}$$

For Equation (D-6) the value of B is defined by the system lg requirement and the downlink rf frequency. The value of $\sigma_{T_e}^{2}\,min$ can be selected on the basis of degradation to E_s/N_o (see Table A). Therefore the value of $\omega_{n\,min}$ can be determined.

For a loop that must operate only in the presence of thermal noise the total mean square tracking error is

$$\sigma_{T_e}{}^2 = \sigma_n{}^2$$

$$= A\omega_n \tag{D-7}$$

where A is defined in Equation (D-1).

Solving Equation (D-7) for ω_n gives

$$\omega_n = \frac{\sigma_{T_e}{}^2}{A} \tag{D-8}$$

The value of A is defined by the data rate and E_s/N_o. The value of $\sigma_{T_e}{}^2$ can be selected on the basis of degradation to E_s/N_o (see Table A). Therefore, the value of ω_n can be determined.

TABLE A

$\sigma_{T_e}{}^2$ (rad^2)	Degradation to E_s/N_o (dB)
0.027	0.1
0.045	0.2
0.066	0.3
0.088	0.4
0.108	0.5
0.129	0.6
0.148	0.7
0.168	0.8
0.187	0.9
0.205	1.0

DERIVATION OF PHASE NOISE SPECIFICATION

In this analysis the following parameters will be used

$B = [1.54 \times 10^3]^2$ for 1g at 7.5 GHz

$\sigma_{T_e}{}^2 = 0.148$ rad^2 for 0.7 dB degradation in E_s/N_o due to carrier tracking

$A = 3.1 \times 10^{-3}$ — for operation at $(E_s/N_o = 2.5$ dB into the Costas Loop $R_s = 150$ bps

$\gamma = 29$ — ratio of data rate to loop corner frequency typical of Phase II, Stage II

The major source of phase noise in Phase II will be flicker of phase produced by the frequency synthesizers.

If the phase noise is restricted to contributing no more than 1% of the total mean square tracking error $(\sigma_{T_e}{}^2)$, we have

$$\sigma_p{}^2 = 0.00148 \text{ rad}^2 \tag{D-9}$$

From Graph 3 for $\gamma = 29$ we find

$$\sigma_p{}^2 = h_1(1.05) \tag{D-10}$$

Combining Equations (D-9) and (D-10) and solving for h_1 gives

$$h_1 = 1.41 \times 10^{-3} \tag{D-11}$$

From Table I of Section 2.4, the power spectral density of the phase noise is

$$S_\varphi(\omega) = \frac{1.41 \times 10^{-3}}{|\omega|} \tag{D-12}$$

or

$$S_\varphi(2\pi f_m) = \frac{2.25 \times 10^{-4}}{f_m} \tag{D-13}$$

Substituting Equation (D-13) into Equation (5a) when B = 1 Hz, the following expression for the phase noise power to carrier power ratio is obtained

$$\frac{N_p}{P} = \frac{2.25 \times 10^{-4}}{|f_m|} + \frac{KT}{C} \tag{D-14}$$

It should be noted that Equation (D-14) is based on a two-sided spectrum.

The value of C/KT should be selected on the basis of the terminal and satellite with which the modulation subsystem is employed. Full satellite power should also be allocated. It should be noted at this point that the satellite, due to the crystals employed, will generate flicker of frequency phase noise. Because of a present lack of information as to the satellite spectral purity, the flicker of phase contributed by the terminals will be considered the major source of phase noise.

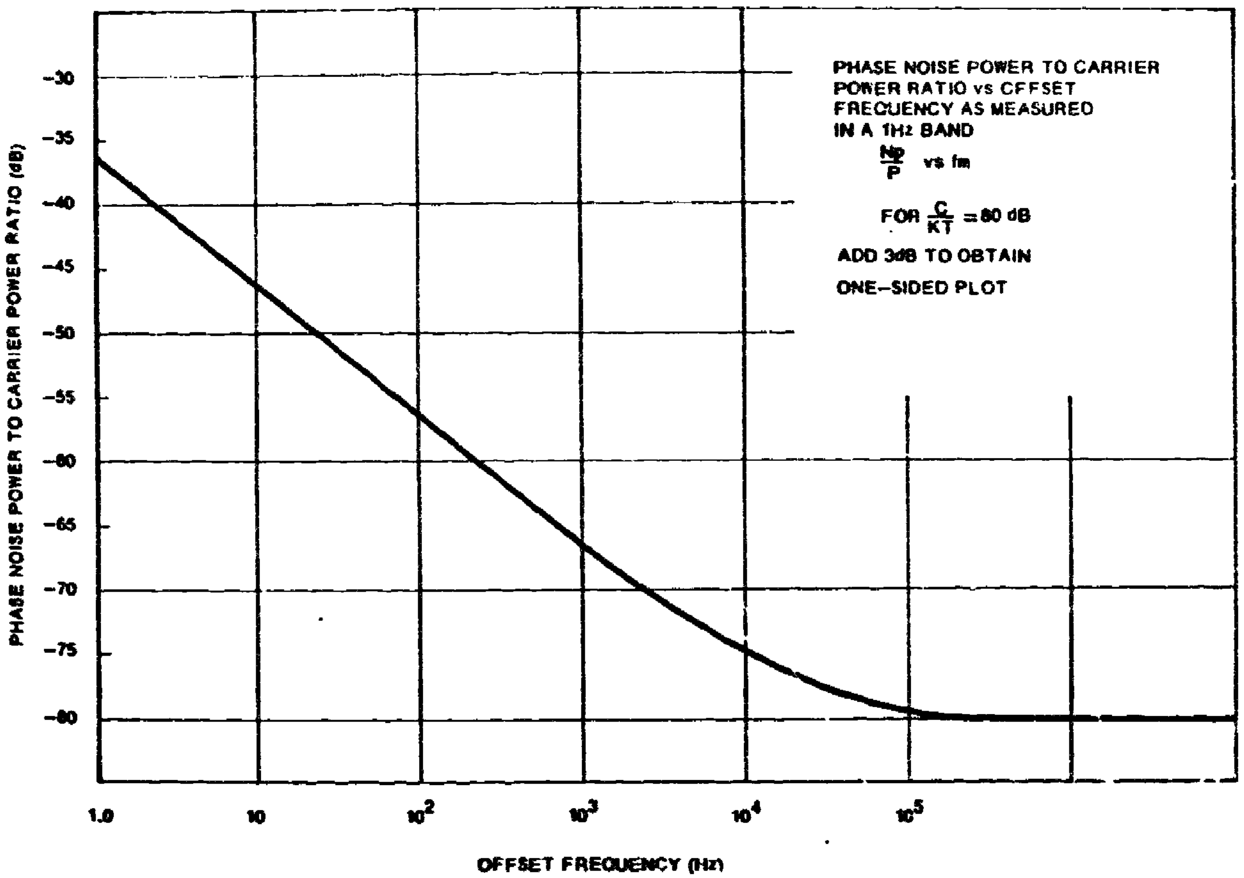
For illustrative purposes a C/KT of 80 dB corresponding to an AN/TSC-54 in the earth coverage channel of the Phase II satellite will be used. Graph D-1 is a plot of the phase noise power-to-carrier power ratio verses offset frequency. The plot is based on the assumptions of this appendix which are listed below.

$\sigma_{T_e}^{\ 2} = 0.148\ \text{rad}^2$	for a 0.7 dB degradation in E_B/N_o due to imperfect carrier regeneration
$\sigma_p^{\ 2} = 0.00148\ \text{rad}^2$	for a 1% contribution to total mean square error by flicker of phase, phase noise
$\gamma = 29$	Data rate to loop corner frequency ratio typical in Phase II Stage II operations
$B = 1$ Hz	Phase noise power is measured in a 1 Hz band

An interpretation of the significance of Graph D-1 is as follows, if the phase noise power to carrier power ratio versus offset frequency curve obtained at the input to the modulation subsystem falls below that of Graph D-1, the phase noise specification shall have been satisfied.

TABLE B

f_m	$\frac{2.25 \times 10^{-4}}{\lvert f_m \rvert} + 10^{-8}$	N_p/P dB
1	2.25×10^{-4}	-36.5
10	2.25×10^{-5}	-46.5
100	2.25×10^{-6}	-56.5
1000	2.35×10^{-7}	-66.3
2000	1.225×10^{-7}	-68.7
4000	6.6×10^{-8}	-71.8
7000	4.2×10^{-8}	-73.8
10,000	3.25×10^{-8}	-74.9
20,000	2.225×10^{-8}	-76.5
40,000	1.56×10^{-8}	-78.1
70,000	1.32×10^{-8}	-78.8
100,000	1.225×10^{-8}	-79
200,000	1.125×10^{-8}	-79.5
400,000	1.056×10^{-8}	-80
700,000	1×10^{-8}	-80


PHASE NOISE POWER TO CARRIER
POWER RATIO vs OFFSET
FREQUENCY AS MEASURED
IN A 1Hz BAND
$\frac{Np}{P}$ vs fm
FOR $\frac{C}{KT}$ = 80 dB
ADD 3dB TO OBTAIN
ONE-SIDED PLOT
PHASE NOISE POWER TO CARRIER POWER RATIO (dB)
-30
-35
-40
-45
-50
-55
-60
-65
-70
-75
-80
1.0
10
10^2
10^3
10^4
10^5
OFFSET FREQUENCY (Hz)

Graph D-1

APPENDIX E

The purpose of this Appendix is to establish some "feel" for how much degradation in E_b/N_o results from a synthesizer with a given frequency stability.

The Hewlett-Packard 5100 B will be used in the example. Figure E-1[11] contains a plot of the SSB signal to Hz BW noise ratio as a function of offset frequency for the HP 5100 B. This plot is equivalent to the upper side band of the phase noise spectral density, $S_\phi(\omega)$. From the graph of Figure E-1, it can be concluded that $S_\phi(\omega)$ is of the form

$$S_\phi(\omega) = \frac{h}{|\omega|} \qquad \text{(E-1)}$$

for offset frequencies less than 10^3 Hz. The value of h can be determined from the plot to be $2\pi \times 10^{-8}$.

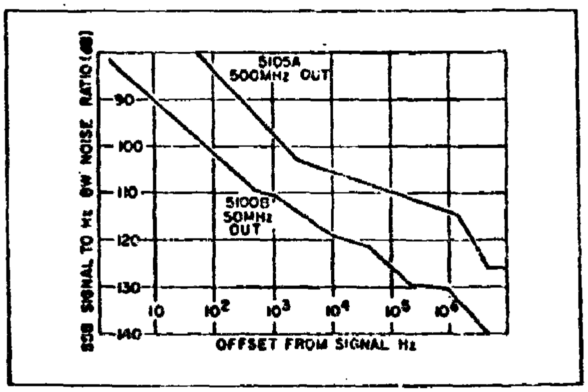


Figure E-1. Composite Phase Noise for Hewlett-Packard Synthesizer

If the 50 MHz 5100B signal were used as a carrier for a PSK modulator, the phase noise would be directly transfered to the PSK signal. Upon demodulation the mean square carrier tracking error due to such phase noise for a data rate to loop corner frequency ratio, γ, of 37.5 can be read directly from Graph 3 of the text.

$$\sigma_p^{\,2} = 2\pi \times 10^{-8}\,(1.14)$$

$$= 7.15 \times 10^{-8} \qquad \text{(E-2)}$$

It should be noted that γ's of 37.5 may be encountered in Phase II modulation subsystems.

The degradation in E_b/N_o due to the variance of Equation E-2 can be calculated using Equation (9) of the text.

$$\begin{aligned} L &= -10 \log_{10} (1 - \sigma_p^{\,2})\ \text{dB} \\ &= -10 \log_{10} (0.9999999285) \\ &\approx -10 \frac{1}{\log_e 10} [-\sigma_p^{\,2}] \qquad \text{(E-3)} \\ L &\approx 3.11 \times 10^{-8}\ \text{dB} \end{aligned}$$

Equation (E-3) implies that the degradation due to phase noise is negligible for a PSK system utilizing the HP 5100 B at 50 MHz. The short term stability of the 5100 B for a 10 ms averaging time, with a 30 kHz noise bandwidth is 6×10^{-10} at 50 MHz.

It should be noted that the preceeding example is not entirely relevant to Phase II Operations. In Phase II terminals the output of frequency synthesizers will have to be converted from MHz to GHz. These conversions will significantly increase the effects of phase noise and long term frequency stability. Consideration will be given to these problems in subsequent work.